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On the Intersection Homology Group of an Analytic Space of Dimension 2

by

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Let X be a compact irreducible analytic space of dimension 2, and $\pi: \tilde{X} \to X$ a resolution of singularities of X. We denote by $b_1(\tilde{X})$ the first Betti number of $\tilde{X}$, by $b_+(\tilde{X})$ the number of positive eigen-values of the intersection product,

$$H_2(\tilde{X}; \mathrm{R}) \times H_2(\tilde{X}; \mathrm{R}) \to \mathrm{R},$$

and by $b_+(X)$ the number of positive eigen-values of the intersection product of the middle intersection homology group of X,

$$IH_2^{\bar{m}}(X; \mathrm{R}) \times IH_2^{\bar{m}}(X; \mathrm{R}) \to \mathrm{R}.$$

In this paper we prove the following theorem.

THEOREM.
(1) $b_1(\tilde{X}) = \dim IH_1^{\bar{m}}(X; \mathrm{R})$,
(2) $b_+(\tilde{X}) = b_+(X)$.

When X is a projective algebraic variety of dimension n, $b_1(\tilde{X}) = \dim IH_1^{\bar{m}}(X; \mathrm{R})$. This is proved by S. Kawai, using the decomposition theorem of P. Deligne, O. Gabber, A. Beilinson, and I. Bernstein (see [1]).

§1.

In this section we assume that X is normal and $\mathfrak{p}$ is a singular point of X. Let $\bigcup_{i=1}^n E_i$ be the decomposition of $\pi^{-1}(\mathfrak{p})$ into irreducible curves. By blowings up of $\tilde{X}$ if necessary, we may assume that
(1) each E_i is non-singular,
(2) if $i \neq j$ and $E_i \cap E_j \neq \varnothing$ then E_i and E_j intersect normally at exactly one point $\mathfrak{p}_{ij}$, which dose not lie on any other E_k,
(3) the function $f(\tilde{x}) = [d(\pi(\tilde{x}), \mathfrak{p})]^2$ is admissible in the sense of D. Mumford [4], where d is the distance of C^n in which some neighbourhood of $\mathfrak{p}$ in X is embedded.

We put $D = \{x \in X \mid d(x, \mathfrak{p}) < \varepsilon\}$, $N = \pi^{-1}(D)$ and $M = \pi^{-1}(\partial D)$, where ∂D is the boundary of D. By [4] if ε is sufficiently small, there exists a continuous map

$\psi: N \to \bigcup_{i=1}^n E_i$ such that ψ gives a deformation retract of N to $\bigcup_{i=1}^n E_i$. Furthermore let $\varphi: M \to \bigcup_{i=1}^n E_i$ be the restriction of ψ to M then it holds that

(1) $\varphi: \varphi^{-1}(E_i - \bigcup_{i \neq j} E_j) \to E_i - \bigcup_{i \neq j} E_j$ is equivalent to the normal S^1-bundle of $E_i - \bigcup_{i \neq j} E_j$ in $\tilde{X}$,

(2) if $\mathfrak{p}_{ij} \in E_i \cap E_j$ then $T_{ij} = \varphi^{-1}(\mathfrak{p}_{ij})$ is homeomorphic to $S^1 \times S^1$,

(3) let U_{ij} be a small disc in E_i with the center $\mathfrak{p}_{ij}$, put $E_i^* = E_i - \bigcup_{i \neq j} U_{ij}$, $M_i^* = \varphi^{-1}(E_i^*)$ and $M_i = \varphi^{-1}(E_i)$, then M_i^* is deformation retract of M_i,

(4) let α_i be a 1-cycle of M_i corresponding to the fibre of M_i over some point in the $E_i - \bigcup_{i \neq j} E_j$ with canonical orientation, and β_{ij} the 1-cycle of T_{ij} which corresponds to α_j by canonical homomorphism $H_1(T_{ij}) \to H_1(M_j) \to H_1(M_j^*)$, then $\langle \beta_{ij} \rangle$ and $\langle \beta_{ji} \rangle$ generate $H_1(T_{ij})$, where $\langle \beta_{ij} \rangle$ is the homology class of β_{ij},

(5) by the homomorphism $H_1(M_i) \to H_1(M_i^*) \to H_1(E_i^*)$, β_{ij} corresponds to $\langle \partial U_{ij} \rangle$.

LEMMA 1. *If ε is sufficiently small,*

(1) $\mathrm{Ker}[H_1(M; \mathrm{R}) \xrightarrow{\varphi_*} H_1(\bigcup_{i=1}^n E_i; \mathrm{R})] = \{0\}$,

(2) $\mathrm{Ker}[H_2(M; \mathrm{R}) \xrightarrow{\varphi_*} H_2(\bigcup_{i=1}^n E_i; \mathrm{R})] = H_2(M; \mathrm{R})$.

Proof. We denote by $H_*(A, B)$ the homology group of the pair (A, B) with coefficient Z.

Sketch of the proof of (1) (*see* [4]). $\mathrm{Ker}[H_1(M_i) \xrightarrow{\varphi_i} H_1(E_i)]$ is generated by $\langle \alpha_i \rangle$ and $\langle \beta_{ij} \rangle$, with only one relation

$$(E_i\, E_i)\langle \alpha_i \rangle + \sum_{i \neq j} (E_i\, E_j)\langle \beta_{ij} \rangle = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (*)$$

where $(E_i\, E_j)$ is the intersection number of E_i and E_j, and $\varphi_i = \varphi_{|M_i}$. By Mayer-Vietoris sequence, we see that $\mathrm{Ker}[H_1(M) \to H_1(\bigcup_{i=1}^n E_i)]$ is generated by $\langle \alpha_1 \rangle, \cdots, \langle \alpha_n \rangle$ with the relations

$$\sum_j (E_i\, E_j)\langle \alpha_j \rangle = 0 \quad (i = 1, \cdots, n).$$

Since the intersection matrix $[(E_i\, E_j)]$ is negative definite (see [3]), $\mathrm{Ker}[H_1(M) \to H_1(\bigcup_{i=1}^n E_i)]$ is finite group. $H_1(\bigcup_{i=1}^n E_i)$ does not contain torsion elements, so

$$\mathrm{Ker}\left[H_1(M; \mathrm{R}) \to H_1\left(\bigcup_{i=1}^n E_i; \mathrm{R}\right) \right] = \{0\}.$$

Proof of (2). For the map of pairs $(M, \bigcup_{i=j} T_{ij}) \to (\bigcup_{i=1}^n E_i, \bigcup_{ij} \{\mathfrak{p}_{ij}\})$, there exists the following commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
\longrightarrow & H_2(M) & \xrightarrow{k_2} & \sum_i H_2(M_i, \bigcup_j T_{ij}) & \xrightarrow{\sum \partial_i} & \sum_i \sum_j H_1(T_{ij}) & \longrightarrow & (\mathrm{ex}) \\
 & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \sum_i \varphi_{i*}} & & \downarrow & & \\
\longrightarrow & H_2(\bigcup_{i=1}^n E_i) & \longrightarrow & \sum_i H_2(E_i, \bigcup_j \{\mathfrak{p}_{ij}\}) & \longrightarrow & 0 & \longrightarrow & (\mathrm{ex})
\end{array}$$

where ∂_i is the connecting homomorphism

$$\partial_i\colon H_2(M_i, \bigcup_j T_{ij}) \to \sum_j H_1(T_{ij}) .$$

By the above diagram it is enough to show that $\operatorname{Ker}\sum \varphi_{i*}\cdot k_2 = H_2(M)$. For $c_i \in H_2(M_i, \bigcup_j T_{ij})$, we write $\partial_i c_i = \sum_j (a_{ij}\langle\beta_{ji}\rangle + b_{ij}\langle\beta_{ij}\rangle)$, where if $E_i \cap E_j = \varnothing$ we let $a_{ij} = 0$ and $b_{ij} = 0$.

(A) $\varphi_{i*}c_i \neq 0$ if and only if $b_{ij} \neq 0$ for some j such that $i \neq j$ and $E_i \cap E_j \neq \varnothing$.

Indeed there is the following commutative diagram with the exact row.

$$\begin{tikzcd}
& & H_2(M_i, \bigcup_j T_{ij}) \arrow[rrr] \arrow[ddd, "\varphi_{i*}"'] \arrow[dr, "\wr"] & & & \sum H_1(T_{ij}) \arrow[dl, "\wr"'] \arrow[ddddl, "\eta"] \\
& & & H_2(M_i, \varphi^{-1}(\bigcup_j U_{ij})) \arrow[r] & \sum_j H_1(\varphi^{-1}(U_{ij})) & \\
& & & H_2(M_i^*, \varphi^{-1}(\bigcup_j \partial U_{ij})) \arrow[u, "\wr"] \arrow[r] & \sum_j H_1(\varphi^{-1}(\partial U_{ij})) \arrow[u, "\wr"'] \arrow[dd] & \\
& & H_2(E_i, \bigcup_j \{p_{ij}\}) \arrow[d, "\wr"] & & & \\
{} \arrow[r] & H_2(E_i^*) \arrow[r] & H_2(E_i^*, \bigcup_j \partial U_{ij}) \arrow[rr, "\xi"] & & \sum_j H_1(\partial U_{ij}) \arrow[r] & (\mathrm{ex})
\end{tikzcd}$$

The positive generator μ of $H_2(E_i^*, \bigcup_j \partial U_{ij})$ corresponds to $\sum\langle\partial U_{ij}\rangle$ by ξ, and $\langle\beta_{ij}\rangle \in H_1(T_{ij})$ corresponds to $\langle\partial U_{ij}\rangle$ by η. So we can easily check (A).

(B) There exists λ_i such that

$$\begin{aligned}
&\left(b_{i1}, \cdots, b_{ii-1}, \sum_j a_{ij}, b_{ii+1}, \cdots, b_{in}\right) \\
&\qquad = \lambda_i((E_i\, E_1), \cdots, (E_i\, E_{i-1}), (E_i\, E_i), (E_i\, E_{i+1}), \cdots, (E_i\, E_n)) .
\end{aligned}$$

Furthermore if $\varphi_{i*}c_i \neq 0$ then $\lambda_i \neq 0$.

Indeed by the following exact sequence

$$\longrightarrow H_2(M_i, \bigcup T_{ij}) \xrightarrow{\partial_i} \sum_j H_1(T_{ij}) \xrightarrow{k_1} H_1(M_i) \longrightarrow (\mathrm{ex}) ,$$

we have $k_1 \cdot \partial c_i = \sum_j (a_{ij}\langle\alpha_i\rangle + b_{ij}\langle\beta_{ij}\rangle) = 0$ in $H_1(M_i)$. So by the relation $(*)$ of the proof of (1), there exists $\lambda_i \in Z$ such that

$$\left(b_{i1},\cdots,b_{ii-1},\sum_j a_{ij},b_{ii+1},\cdots,b_{in}\right)$$
$$=\lambda_i((E_i\,E_1),\cdots,(E_i\,E_{i-1}),(E_i\,E_i),(E_i\,E_{i+1}),\cdots,(E_i\,E_n)).$$

By (A) if $\varphi_{i*}c_i\neq 0$ then there exists j such that $b_{ij}\neq 0$ and $(E_i\,E_j)\neq 0$, so $\lambda_i\neq 0$.

(C) Assume that there exists $(c_1,\cdots,c_n)\in \text{Image}\,k_2$ such that $\sum_i \varphi_{i*}(c_1,\cdots,c_n)\neq 0$. Since $\sum \partial_i(c_1,\cdots,c_n)=\sum_{ij}(a_{ij}+b_{ji})\langle\beta_{ji}\rangle=0$, we have $a_{ij}+b_{ji}=0$ $(i,j=1,\cdots,n)$. So

$$b_{1i}+\cdots+b_{i-1i}+\sum_j a_{ij}+b_{i+1i}+\cdots+b_{ni}=0\quad(i=1,\cdots,n).$$

Hence by (B)

$$\begin{pmatrix}(E_1\;E_1)\cdots(E_n\;E_1)\\(E_1\;E_n)\cdots(E_n\;E_n)\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Since $\sum\varphi_{i*}(c_1,\cdots,c_n)\neq 0$, there exists some c_i such that $\varphi_{i*}(c_i)\neq 0$. So by (B), $\lambda_i\neq 0$. This contradicts the fact that the intersection matrix $[(E_i\,E_j)]$ is negative-definite. Hence $\text{Ker}\sum\varphi_{i*}\cdot k_2=H_2(M)$.

§2.

LEMMA 2. *Let X be a compact normal irreducible complex analytic space of dimension 2, and $\Sigma_X=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_m\}$ the set of singular-points of X. Then*

(1) $IH_1^{\bar m}(X)\simeq H_1(X-\Sigma_X)$,

(2) *let $\omega\colon IH_2^{\bar m}(X)\to H_2(X)$ be the homomorphism induced from the inclusion homomorphism $IC_*^{\bar m}(X)\to C_*(X)$. Then* $\text{Image}\,\omega=\text{Image}[H_2(X-\Sigma_X)\xrightarrow{i_*}H_2(X)]$ *and ω is an isomorphism of $IH_2^{\bar m}(X)$ to* $\text{Image}[H_2(X-\Sigma_X)\xrightarrow{i_*}H_2(X)]$, *where $i\colon X-\Sigma_X\to X$ is the inclusion map.*

Proof. Let T be a triangulation of X such that each $\{\mathfrak{p}_i\}$ is a 0-simplex of T, and T'' be the second barycentric subdivision of T. We put

$$\hat X=X-\bigcup_{i=1}^{n}\{\text{open star of }\mathfrak{p}_i\text{ with respect to }T''\}.$$

Since $\hat X$ is deformation retract of $X-\Sigma_X$, $H_*(\hat X)\simeq H_*(X-\Sigma_X)$ and

$$\text{Image}[H_2(X-\Sigma_X)\to H_2(X)]=\text{Image}[H_2(\hat X)\to H_2(X)].$$

By the definition of the intersection homology group (see [2]), we have $H_1(\hat X)\simeq IH_1^{\bar m}(X)$ and ω is an isomorphism of $IH_2^{\bar m}(X)$ to $\text{Image}[H_2(\hat X)\to H_2(X)]$.

Proof of the theorem. Let $g\colon\bar X\to X$ be the normalization of X, $\pi\colon\tilde X\to\bar X$ a resolution of singularities of $\bar X$ and $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_m\}$ the set of singular points of $\bar X$. Then

g is the normalization in the sense of [2].

Proof of (1). Since $IH_1^{\bar{m}}(X; \mathrm{R}) \simeq IH_1^{\bar{m}}(\bar{X}; \mathrm{R}) \simeq H_1(\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}; \mathrm{R})$, it is sufficient to show that $\dim H_1(\tilde{X}; \mathrm{R}) = \dim H_1(\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}; \mathrm{R})$. Put $E = \pi^{-1}(\{\mathrm{p}_1, \cdots, \mathrm{p}_m\})$, then $\tilde{X}-E$ is homeomorphic to $\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}$. So $\dim H_1(\tilde{X}-E; \mathrm{R}) = \dim H_1(\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}; \mathrm{R})$. Since $\operatorname{codim}_R E > 2$, $H_1(\tilde{X}-E; \mathrm{R}) \to H_1(\tilde{X}; \mathrm{R})$ is a surjection. Hence $\dim H_1(\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}; \mathrm{R}) = \dim H_1(\tilde{X}-E; \mathrm{R}) \geq \dim H_1(\tilde{X}; \mathrm{R})$.

By blowings up of $\tilde{X}$ if necessary, we may assume that each component of E satisfies the conditions of § 1. As § 1 we define N, M, $\psi: N \to E$, and $\varphi: M \to E$.

The following commutative diagram with exact rows exists.

$$\begin{array}{ccccccc}
\longrightarrow H_2(\tilde{X}, \tilde{X}-E; \mathrm{R}) & \longrightarrow & H_1(\tilde{X}-E; \mathrm{R}) & \xrightarrow{i_*} & H_1(\tilde{X}; \mathrm{R}) & \longrightarrow & (\mathrm{ex}) \\
\uparrow \wr & & \uparrow & & \uparrow & & \\
\longrightarrow H_2(N, M; \mathrm{R}) & \longrightarrow & H_1(M; \mathrm{R}) & \xrightarrow{\varphi_*} & H_1(E; \mathrm{R}) & \longrightarrow & (\mathrm{ex})
\end{array}$$

By (1) of Lemma 1, $\operatorname{Ker} \varphi_* = \{0\}$. So i_* is an injection. Hence

$$\dim H_1(\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}; \mathrm{R}) = \dim H_1(\tilde{X}-E; \mathrm{R}) \leq \dim H_1(\tilde{X}; \mathrm{R}).$$

Proof of (2). By Mayer-Vietoris sequence, we have the following exact sequence of homology groups with coefficient R,

$$\begin{aligned}
&\longrightarrow H_2(M) \longrightarrow H_2(\tilde{X}-E) \oplus H_2(E) \longrightarrow H_2(\tilde{X}) \\
&\qquad \longrightarrow H_1(M) \xrightarrow{(j_*, \varphi_*)} H_1(\tilde{X}-E) \oplus H_1(E) \longrightarrow (\mathrm{ex})
\end{aligned}$$

where j is the inclusion map $j: M \to \tilde{X}-E$.

Since $\operatorname{Ker}[H_1(M) \xrightarrow{\varphi_*} H_1(E)] = \{0\}$, we have $\operatorname{Ker}(j_*, \varphi_*) = \{0\}$. Hence $H_2(\tilde{X})$ is generated by $\operatorname{Image}[H_2(\tilde{X}-E) \to H_2(\tilde{X})]$ and $H_2(E)$. Put $K = \operatorname{Image}[H_2(\tilde{X}-E) \to H_2(\tilde{X})]$, $L = \operatorname{Image}[H_2(\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}) \to H_2(\bar{X})]$. By Mayer-Vietoris sequence, we have the following commutative diagram with exact rows,

$$\begin{array}{ccccccc}
\longrightarrow H_2(M) & \longrightarrow & H_2(\tilde{X}-E) \oplus H_2(E) & \longrightarrow & H_2(\tilde{X}) & \longrightarrow & (\mathrm{ex}) \\
\downarrow & & \wr\downarrow \quad \downarrow & & \downarrow & & \\
\longrightarrow H_2(M) & \longrightarrow & H_2(\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}) \oplus H_2(\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}) & \longrightarrow & H_2(\bar{X}) & \longrightarrow & (\mathrm{ex})
\end{array}$$

Since $\operatorname{Ker}[H_2(M) \to H_2(E)] = H_2(M)$ and $H_2(\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}) = \{0\}$, π_* is an isomorphism of K to L. The intersection product of K as subspace of $H_2(\tilde{X})$ and that of L as $IH_2^{\bar{m}}(\bar{X})$ are determined by the cycles in $\bar{X}-\{\mathrm{p}_1, \cdots, \mathrm{p}_m\}$ which is homeomorphic to $\tilde{X}-E$ by π, in the same manner. So if we identify K with L by π_*, these two products are the same. Since $(\tilde{X}-E) \cap E = \varnothing$, K is the orthogonal complement of $H_2(E)$ in $H_2(\tilde{X})$ with respect to the intersection product of $H_2(\tilde{X})$. Since the intersection matrix $[(E_i E_j)]$ is negative definite, each of the eigen-values of the intersection product

$H_2(E)\times H_2(E)\to \mathrm{R}$ which is induced from the intersection product of $H_2(\tilde{X})$ is negative. Hence $b_+(\tilde{X})=$[the number of positive eigen-values of the intersection product of K]$=$[the number of positive eigen-values of the intersection product of L in $IH_2^{\bar{m}}(\bar{X})$]$=b_+(\bar{X})$. By the definition of intersection product, we see that the isomorphism $IH_2^{\bar{m}}(\bar{X})$ to $IH_2^{\bar{m}}(X)$ induced from the normalization $g: \bar{X}\to X$ dose not change the intersection product. So $b_+(\bar{X})=b_+(X)$. Hence $b_+(\tilde{X})=b_+(X)$.

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